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Proposed solution of the quantum measurement problem

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Abstract. A discussion of quantum measurement theory is presented. It is argued that the quantum measurement problem consists in the failure of any formalism based on unitary evolution to describe fully the observed behaviour in measurement-like processes. It is shown that the problem is essentially concerned with the phase of the coherence between orthogonal eigenstates of an object/measuring device system. It is then proposed that the observed behaviour is fully described if the value of this phase is formally undecidable.

1. Introduction—the nature of the problem

The problem in the basic workings of quantum mechanics can be stated as follows: the behaviour of many physical systems is not fully described by Schrödinger's equation (or its equivalents). Schrödinger's equation always produces a unitary evolution, while systems such as measuring devices are frequently involved in non-unitary changes in their state [1]. An illustrative example is given in the appendix. This problem is not one of philosophy or interpretation, but one of physics and mathematics, or so we will argue below.

It is common practice to avoid stating the problem in such stark terms. One can follow the detailed quantum description of a measuring apparatus, and it soon emerges that coherences between parts of the apparatus' complete quantum state become vanishingly small (or have rapidly varying phase). The parts which thus lose their mutual coherence can be recognised as representing different macroscopic situations. At this point one merely states that in practice the apparatus will adopt (on a random basis) just one of the distinguishable possibilities. It is not that the quantum calculations have failed to show how the apparatus actually reaches its final state; rather the only meaning of the quantum mechanics was as a tool to supply which final states the apparatus might adopt, and their relative probabilities.

Arguments like that of the previous paragraph fail to resolve the problem. To say that the quantum calculations are a tool to be used to obtain the possible states of a macroscopic system, one first needs to identify one's macroscopic system. However, there is no prescription for telling whether a given apparatus is such a macroscopic system, or is merely part of the quantum tool to be used to describe another system. This produces the problem of the 'chain of measuring devices' [2]: where do you draw the line between the quantum world with its superpositions, and the definiteness of a completed measurement? (i.e. where or how does nature draw such a line).

In section 2 we will summarise the mathematical description of a measurement process, and show where the description is incomplete. In section 3 we will argue that a certain minimum requirement may be made of the evolution of a quantum system,

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which, if it came about, would be sufficient to solve the measurement problem. In section 4 we will propose a means by which the required condition may come about.

2. Quantum measurement theory

There are three main ingredients in the complete description of a measurement process. Firstly, the object being measured should interact with a measuring device in such a way as to produce a correlation between states of the measuring device and states of the object. For example, let the measured object O be a two-state system such as a single spin, spanned by the basis states $|+\rangle$, $|-\rangle$. Let the object be in interaction with another two-state system M spanned by $|2\rangle$, $|1\rangle$. O and M are treated in quantum mechanics as a composite four-state system, the basis states of which may be conveniently written as

$$|+\rangle|1\rangle \rightarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |-\rangle|1\rangle \rightarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |+\rangle|2\rangle \rightarrow \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |-\rangle|2\rangle \rightarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{1}$$

Initially, system O is in the state $(\alpha|+\rangle + \beta|-\rangle)$, and system M is in the state $|1\rangle$, so that the composite state is

$$I = \begin{pmatrix}\alpha\\ \beta\\ 0\\ 0\end{pmatrix}.$$

The density matrix for this (pure) state is

$$\begin{pmatrix}|\alpha|^2 & \alpha\beta^* & 0 & 0\\ \alpha^*\beta & |\beta|^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

where * denotes complex conjugation.

Let the evolution of the system be governed by the propagator

$$U = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & \exp(-\mathrm{i}\delta)\\ 0 & 0 & 1 & 0\\ 0 & \exp(\mathrm{i}\delta) & 0 & 0\end{pmatrix}. \tag{2}$$

Then the final state of the system is UI, with density matrix

$$F = \begin{pmatrix}|\alpha|^2 & 0 & 0 & \alpha\beta^*\exp(-\mathrm{i}\delta)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \alpha^*\beta\exp(\mathrm{i}\delta) & 0 & 0 & |\beta|^2\end{pmatrix}. \tag{3}$$

The axioms of quantum mechanics require that the propagator U must be unitary, as it is here. For $\delta = 0$, U may be derived from the Hamiltonian [3]

$$H = g(t)(1-\sigma_z)P \qquad U = \exp\left(-\mathrm{i}\int H\, \mathrm{d}t/\hbar\right)$$

where σ_z is the z Pauli spin matrix, operating on O, and P is the projection operator

$$P = \tfrac{1}{2}(\,|1\rangle - |2\rangle\,)(\,\langle 1| - \langle 2|\,).$$

In the final state F there is the required correlation between states of the object O and measuring device M.

The next ingredient in the description of the measurement process is that the off-diagonal elements of the density matrix F should vanish. The basis (of object states) in which this occurs is the set of states which are eigenstates of some observable: this is the observable which the device is said to be measuring. The diagonalisation of the density matrix describes the fact that after the measurement no effects can be observed which depend on interference between the parts $|+\rangle$ and $|-\rangle$ of the original state of the object.

There are two ways in which a completely unitary evolution can lead to an approximately diagonal density matrix, as shown by Haake and Walls [4]. Firstly, one can argue that a two-state measuring device is too great a simplification, and one employs instead a many-state or even continuous 'pointer variable'. The final state of the O + M system can then be a superposition for which the pointer state consists of two parts with essentially no overlap between them. The relevant off-diagonal density matrix elements (coherences) are then infinitesimally small. Secondly, instead of being infinitesimal, the coherence might merely have a phase which varies very rapidly with the value of the pointer variable. If the system is subject to a small amount of noise, the coherence vanishes due to the rapid variation of the phase.

We will now argue that the more general case is that for which the coherence has a highly sensitive phase. For any multistate measuring device M′, one can return to the two-state measuring device M described in (1) as follows: simply add to M′ a device which automatically prepares a two-state quantum system in one of its states, depending on the value of the pointer variable of M′. For example, the GM tubes discussed in the appendix may be linked to a Penning trap so that a single trapped electron can be rotated into the spin-up or spin-down direction, with respect to a fixed axis, depending on which tube fires. One can then regard the trapped electron as the measuring device M which interacts with O, the effects of the trap and M′ being incorporated into the propagator (2). With a two-state measuring apparatus, the coherences in F cannot vanish or even become small, but they can have a phase δ which is highly sensitive to perturbations and to changes in the initial conditions.

Many authors are content to finish the discussion once an approximately diagonal density matrix is obtained. It is then argued that the density matrix describes the statistics of the O + M system, and that the only way in which an ensemble of systems can have zero coherence is that each individual system should adopt one of the states

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

(with probabilities given by the diagonal elements of F). However, the state F is a perfectly respectable quantum state. So far the description has given no reason why the final state of each system in an ensemble should not be given by (3), with various values for the phase δ. The difference between (3) and (4) is essentially the quantum measurement problem.

The final ingredient in the description of a quantum measurement process is, then, some means by which the mathematics should unambiguously show that the final state is given not by (3) but by (4). The textbook approach is usually to state by postulate that the system is described by (4) 'after a measurement has occurred', the transition from (3) to (4) being called 'the collapse of the wavefunction' (but note that this phrase is used differently in [4]). However, exactly what constitutes a measurement is not specified.

Two further comments should be made. Firstly, the situation given by (4) cannot be arrived at by unitary evolution. Secondly, the two possible final states in (4) usually correspond to differences in the world lines of the relevant systems over an extended region of spacetime. The latter implies that it may not be meaningful to identify a time 'at which' the state 'collapses' from (3) to (4). This is an example of the non-local character of the quantum theory [2].

3. A minimum requirement

So far, the mathematical description (up to equation (3)) provides a method by which a physicist, looking at the equations, can decide where a non-unitary evolution will occur: simply look for coherences with extremely sensitive phase. The possible final states at the end of the non-unitary evolution are also clear (those states whose mutual coherences have obtained a very sensitive phase), as are their relative probabilities. However, as yet there is no mathematical reason why the quantum/measuring-device system should not remain in a superposition state. The phase of the coherence may be rapidly changing, but in principle it has some value (even if on repeated runs of the experiment it is technologically impossible to reproduce a given value and thus observe the coherence via an interference effect.) To solve this dilemma, the minimum requirement we need make of the evolution of the quantum/measuring-device system, is, we will argue, as follows.

(R) In a process of the measurement type, the phase of the coherence (between orthogonal eigenstates of the measured observable) acquires an additional property (which we leave unspecified for the moment) which makes it mathematically distinguishable from phases arising in normal, non-measurement-type, evolution.

The idea behind this requirement is that we try to include actually in the mathematics the job which previously the physicist had to do by means of intuition. A candidate for the 'additional property' mentioned in requirement R is given in section 4. The property which we have already noted, that of being highly sensitive to initial conditions and perturbations, is not sufficiently distinctive for our purposes, since the degree of sensitivity depends on what time or distance scale one is considering. We will, however, return to this possibility in section 4.

We now need to show how requirement R is sufficient to solve the measurement problem. At this point in the argument, one might imagine a quantum system whose evolution, as determined by Schrödinger's equation, puts it in the state

$$|S\rangle = (1/\sqrt{2})(|+\rangle + \exp(i\delta)|-\rangle)$$

where the phase δ satisfies R. Since we have a measuring-type process, however, the final state of the system will actually be either $|+\rangle$ or $|-\rangle$, and not the superposition given by $|S\rangle$. To make the theory correctly describe this experimental fact, we state as a basic postulate of the theory:

Postulate 1. The state vector of a quantum system evolves according to Schrödinger's equation. The physical meaning of a state containing phases with the special property mentioned in R is that the system will evolve into one or other of the states thereby singled out.

This postulate replaces the usual postulate concerning the state of a quantum system after a 'measurement' has occurred. Postulate 1 is an improvement because the former postulate failed to define what is meant by a 'measurement'. Note that the loss of coherence information brings irreversibility into the dynamics: this is similar to Boltzmann's proposal for resolving the paradox of how irreversible macroscopic behaviour can arise from apparently reversible microscopic equations.

The argument relies on the idea that the coherence phase can single out one set of basis states from all the other sets. This is not obvious, since if δ is allowed to vary, then interference effects will be 'washed out', no matter which basis one uses to specify the final state. However, the basis given by (1) is the only one in which the coherence $\alpha\beta^* \exp(-i\delta)$ describes a circle in the complex plane, as δ varies. The argument can be applied to a many-state system also—one examines the coherence between each pair of substates of the system, choosing each sub-basis accordingly.

4. Proposed means by which the measurement problem is solved

The argument of section 3 is only convincing if we can find an acceptable candidate for the special 'additional property' mentioned in R. We will make a few observations, and then propose such a candidate.

We have already noted that Schrödinger's equation can lead to coherence phases which are very sensitive to initial conditions and to perturbations. We made the comment that this property does not solve the measurement problem, since although the phase in this case may be rapidly varying, it still has some definite value and the coherence is, in principle, just as definite as that arising in more simple evolution. However, what if the phase were so sensitive that there is *no* time- or distance-scale on which the phase is not rapidly varying? This is just the kind of behaviour found in classical systems having the property of 'mixing'. (In 'mixing' systems two or more types of motion are possible, and points infinitesimally close together in any region of the system's phase space belong to different types of motion. This is closely related to classical chaos.) The avalanche or other complicated processes occurring in measuring devices are all likely to have mixing or similar properties. However, mixing is suppressed in quantum systems [5, 6]. Structure in phase space does not occur on a scale smaller than $\hbar$, so that all the features of the quantum state, including δ, can be accurately described.

Peres' discussions [3, 7] may be summarised as the view that the distinctive property mentioned in R is this: the coherence phases are randomised by external influences. These influences are, of course, due to other physical systems, which have not been included in the quantum theory of the system under investigation. He argues that just as it is impossible in practice for a system to be completely isolated from its surroundings, so also it is part of the nature of the quantum theory that something must remain unanalysed. Any given measuring device, for instance, can be analysed; however, one will always have small effects remaining which are left out of the analysis but which are sufficient to randomise the phase. This point of view would solve the problem in the vast majority of cases. It falls down eventually, however, when the evolving physical system is the whole universe, which presumably does behave as an isolated system.

How easy might it be to calculate the value of a highly sensitive coherence phase? Such a calculation must be made by means of physical processes such as writing on paper, electronic computing, or perhaps merely the firing of many neurons. If the phase were sufficiently sensitive, however, these physical processes would themselves have a significant influence on the value of the phase. This might cause such phases to be impossible to calculate.

This leads us on to the following idea: can we envisage that the universe itself cannot 'calculate' the value of the phase?—i.e. the phase has no definite value? This rather unclear proposal can be given a more firm expression, by means of Gödel's theorem [8]. Gödel's theorem states, briefly, that in any 'formal system' of sufficient complexity, it is possible to state a theorem, within the syntax of the formal system, whose truth value cannot be determined within that formal system. Such theorems are said to be 'formally undecidable'. If, as we propose, the physical universe may be said to be such a formal system, then there exist physical systems whose behaviour is undecidable within the formal system of the universe. That is, one could find a property of such a system which is not merely incomputable in practice, but which could be proved to be formally undecidable.

We propose that the measurement problem be solved as follows.

Postulate 2. The Schrödinger equation of a measurement-type process is Gödelian. Specifically, the coherence phase δ, which is subject to the Schrödinger equation, has the property that there exists a value θ for which the proposition '$\theta - \pi < \delta < \theta + \pi$' is formally undecidable.

We now have a means by which requirement R may be satisfied, and so the measurement problem is solved, via postulate 1. The undecidable proposition can be simplified as follows: first quantise the phase δ by associating with it an integer n given by

$$\delta \leqslant n\pi < \delta + \pi.$$

Postulate 2 then states that it is undecidable whether the solution to the Schrödinger equation has odd or even n.

Kanter [9] has recently discussed undecidable correlations arising in classical systems. It is well known that physical systems can have undecidable behaviour. For example, a digital computer can be programmed to search methodically for the solution to an undecidable problem—it is then undecidable as to whether the computer program will ever halt. Kanter shows how to identify an undecidable feature of the state of a very simple and general class of systems. However, the undecidability in all these cases is related to the boundedness of the system: as soon as the system has a finite number of possible states, its behaviour is decidable. For example, a finite computer (or Ising system) can only propose trial solutions to the 'domino-snake' problem [9] with a finite number of tiles. Quantum systems (of finite size, energy etc) cannot have infinitely many states, unlike their classical counterparts, so that it is not straightforward to identify truly undecidable physical systems. However, if we return to the idea that the physical universe itself may be regarded as a formal system, then it is reasonable to suppose that undecidable behaviour is possible within it.

Finally, the question may be asked as to exactly how or when the O+M system evolves from equation (3) to equation (4). The answer is that one need not attempt to specify this. The mathematical description gives all the possible information about the experimental situation, including the information as to whether the final state will be (3) or (4). In making calculations, one can in fact use a state of the type (3) *or* (4) to describe the system at *any* time (i.e. even before the measurement process has occurred), as Aharonov and Vaidman [10] point out. The undecidable phase simply serves as an indicator that one must expect the experimental apparatus to adopt one of the states given by (4).

5. Conclusion

In this paper we have argued that the measurement problem is a problem in mathematics rather than physical interpretation, involving the failure of the quantum theory to describe adequately the non-unitary processes which occur in the real world. In section 3 we identified where in the mathematics a new feature needs to be introduced, and argued that if the coherence phase in a measurement-type process were somehow of a different nature to that rising in more simple processes, then the non-unitary evolution would be adequately described. This is so because once the coherence phase is identifiable, then so are the possible final states at the end of the non-unitary evolution, and one can merely state by postulate that the system is put into one of these states.

In section 4 we reviewed some ideas as to how the required behaviour for the phase might come about in practice. We finally proposed that the special property of the coherence phase is that its value is formally undecidable within the formal system of the universe. This proposal appears here in a somewhat speculative fashion. The evidence for its validity is that the thrust of the nature of the problem seems to be pointing towards this type of resolution. To describe a measurement process fully, one must include the effects of a wider and wider 'all-embracing' system, as each successive attempt to include all influences makes the behaviour sensitive to smaller and smaller as-yet-unincluded parts of the 'environment'. This implies that a complete description may need to be self-referential, in order that the universe may completely specify its own state.

The present argument at least suggests that a worthwhile programme of research would be to try to find a system which is described by a Gödelian Schrödinger equation. It may be possible to approach this task in a general way by extending the phase-space mapping ideas used in the study of chaos. If the system could be regarded as a model for a measurement process, and if it could be argued that its undecidable feature can be pinpointed as the coherence phase, then one would feel that a suitable and complete model of non-unitary (measurement-like) processes had been found.

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Appendix. Illustration of the measurement problem

The gap in basic quantum mechanics which we are referring to as the measurement problem may be illustrated as follows.

It is not clear how to give a detailed description of exactly what goes on when, for example, a single electron in a quantum superposition state is able to enter two Geiger-Muller (GM) tubes simultaneously. (We use a Geiger-Muller tube since it is easy to describe, but the argument applies in principle to any measuring device, or to similar processes even if they are not normally referred to as a 'measurement'.)

The process occurring in a Geiger-Muller tube is, roughly, that a particle entering is scattered by gas molecules in the tube. The collisions ionise some of the molecules. The ions are accelerated in the high electric field of the tube and produce more ions in further collisions, until an avalanche of ions and electrons is registered as a pulse of current in the tube.

Now consider an electron in a quantum state which has a finite amplitude for the electron to pass into two separate tubes A and B (for example, the tubes are positioned at the output of a Stern-Gerlach apparatus, or at the slits in a Young's slits experiment, etc). Initially, the electron's state is a superposition of 'electron entering A' and 'electron entering B'. As the electron 'enters' the two tubes, the first processes that occur are quantum scattering off one or two gas molecules. The state of the electron/gas-molecule system becomes a superposition of 'electron in A, gas molecules in A scattered' and 'electron in B, gas molecules in B scattered'. The electron and scattered molecules (or ions) then produce further collisions, but every such collision is a quantum process, so that the state of the whole electron/GM-tube system remains a superposition of 'electron in A, gas molecules in A scattered' and 'electron in B, gas molecules in B scattered'. Thus the 'measuring instrument' cannot perform the measurement: its final state consists of a superposition of the states corresponding to all the possible outcomes of the measurement. In practice, however, one of the GM tubes certainly registers a current pulse (or would if it had a higher quantum efficiency), and the other certainly does not, so we have a paradox—it is the Schrödinger cat paradox [2]. Further comments can be made, such as that a higher-order measuring instrument (such as a conscious observer) could measure the state of the GM tubes, and that as long as the superposition is eventually collapsed, the physical predictions are independent of the stage at which the collapse occurs. However, it should be possible to understand the working of a simple Geiger-Muller tube without resort to undescribed 'higher-order' systems: the attempt to understand it should at least be made.

The point made above may be expressed mathematically by noting that Schrödinger's equation or its equivalents always produces a unitary evolution of the state vector, so that if any part of a composite system (here the electron) is in a superposition, then the whole composite system (electron plus GM tubes) is in a superposition. The fact that the final state of the GM tubes is not a superposition, in any given experiment, implies that the evolution must have been non-unitary, and therefore not described by Schrödinger's equation. This argument applies even if the final superposition state has a nearly diagonal density matrix and so obeys classical statistics, as discussed in section 2 above.

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